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Ternary relations and relevant semantics

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Abstract

Modus ponens provides the central theme. There are laws, of the form $A \rightarrow C$. A logic (or other theory) L collects such laws. Any datum A (or theory T incorporating such data) provides input to the laws of L. The central ternary relation R relates theories L, T and U, where U consists of all of the outputs C got by applying modus ponens to major premises from L and minor premises from T. Underlying this relation is a modus ponens product (or fusion) operation $\circ$ on theories (or other collections of formulas) L and T, whence $RLTU$ iff $L \circ T \subseteq U$. These ideas have been expressed, especially with Routley, as (Kripke style) worlds semantics for relevant and other substructural logics.

Worlds are best demythologized as theories, subject to truth-functional and other constraints. The chief constraint is that theories are taken as closed under logical entailment, which clearly begs the question if we are using the semantics to determine which theory L is Logic itself. Instead we draw the modal logicians' conclusion—there are many substructural logics, each with its appropriate ternary relational postulates.

Each logic L gives rise to a Calculus of L-theories, on which particular candidate logical axioms have the combinatorial properties expected from the well-known Curry–Howard isomorphism (with improvements by Dezani and her fellow intersection type theorists.). We apply their bubbling lemma, updating the Fools Model of Combinatory Logic at the pure $\rightarrow$ level for the system $\mathbf{B}^\wedge\mathbf{T}$. We make fusion $\circ$ an explicit connective, proving a combinator correspondence theorem. Having taken relevant $\rightarrow$ as a left residual for $\circ$, we explore its right residual mate $\rightarrow r$. Finally we concentrate on and prove a finite model property for the classical minimal relevant logic **CB**, a conservative extension of the minimal positive relevant logic **B**+.

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0. Introduction

Logic is the science of rational inference. Founded by Aristotle, it has through the introduction of mathematical methods made remarkable progress in the past century and a half. Polish logicians—headed by the great Alfred Tarski, to whose memory this paper is dedicated—have played a leading role in this progress.

This paper surveys some ideas of mathematical beauty, which extend Tarski's legacy. These ideas have resulted from three decades of investigation of the semantics of relevant and other substructural logics. Just as modal logics form a *family*, so also do the relevant logics. Just as there is a minimal normal modal logic **K**, just so is there a minimal classical relevant logic **CB**. Just as modal logics may be motivated via the relational properties of a *binary* accessibility relation, so can relevant logics via the properties of a *ternary* relation R. Of special interest is the *correspondence* between candidate axioms for relevant logics and the *combinators* of Curry's Combinatory Logic **CL** and Church's calculi $\boldsymbol{\lambda}$ of λ-conversion.

This combinatorial *Key to the Universe* will be a main topic of this paper, whose purpose (under the incisive prodding of a referee) will be chiefly *expository*. We shall begin with a very brief survey of relevant logics—where they came from, and what are the central points of their formal and philosophical motivation. A regard for *relevance* will boil down to a careful look at the rule $\to E$ of *modus ponens*. That look will lead us on to *theories*, collections of sentences that are *closed under entailment* (in the appropriate sense).

1. Relevant logics

When does *A imply C*? The simple material answer, in *practical* vogue toward the end of the last millennium, is that this happens iff either A is *false* or C is *true*. But this answer is so deeply unsatisfying *philosophically* that almost nobody believes it. (Belnap observes that Bertrand Russell *might have*, for a while.) *At least*, most would concede, A must be in some sense *honestly related to C* for a genuine implication to hold.

1.1. Church's use criterion for relevant implication

A well-known proposal—essentially that of [7] for pure $\to$ logic—*tracks* this sense of relevance. $A \to C$ *holds* (and we may say that *A relevantly implies C*) iff there exists a *deduction* of C from A in which A is *used*. More generally (and associating $\to$ to the *right*) $A_1 \to \cdots \to A_n \to C$ holds iff there is a deduction of C from $A_1, \ldots, A_n$ in which *all* of the A_i are used.

This is the *use criterion* for separating the relevantly valid, pure $\to$ *sheep* from irrelevant *goats* like the positive paradox $p \to (q \to p)$ of material implication. This use criterion is discussed at some length in the standard treatise [2]. Supplied there with a Fitch-style natural deduction system, it is taken to motivate *exactly* the *weak theory* $\mathbf{R}\!\to$ of (*pure relevant*) *implication* of Church [7].

Central to the application of the use criterion is that 'use' *means*, at the pure $\rightarrow$ level, 'use in an application of the rule $\rightarrow E$ of *modus ponens* for $\rightarrow$'. Also stressed is that it is *sufficient* for the relevant theoremhood of $A_1 \rightarrow \cdots \rightarrow A_n \rightarrow C$ that *there exist* a deduction of C in which all the A_i are used.

There are some subtleties in how this use criterion may be varied to produce different systems of logic, even at the pure $\rightarrow$ level. See [20].

1.2. Adding extensional connectives to relevant $\rightarrow$

The first relevant logic to build in recognizable truth-functional connectives and quantifiers was the *strenge Implikation* of Ackermann [1], which we identify here with the **E** (of entailment) of Anderson and Belnap [2] and the $\mathbf{E}^{\forall\exists x}$ (of first-order entailment) of Anderson et al. [3]. These systems add $\wedge, \vee$ and $\sim$ at the propositional level, intended as truth-functional 'and', 'or' and 'not'; roughly speaking, $\wedge$ and $\vee$ are *distributive lattice connectives*, and *duals* relative to the (DeMorgan) negation $\sim$. Similarly enriching $\mathbf{R}\rightarrow$ produces a full propositional logic **R**.

What, in the presence of these truth-functional connectives, becomes of the use criterion? Its care and feeding become in practice more delicate. Suffice it to say here that the composition of premisses under $\rightarrow E$ must be supplemented by attending to their composition under the rule $\wedge I$ of *conjunction introduction*. I shall not, at this point, go into technical details, save to note that the result remains mathematically coherent.

1.3. Bunching premisses relevantly

Well, I will go into a *few* technical details, about how premisses are *bunched* for relevantly valid argument. We saw that, at the pure $\rightarrow$ level, it must be possible to use *all* the hypotheses of an argument for the conclusion to have been derived *relevantly*. There is a relevant connective that does this bunching, namely the *fusion* (or *intensional* conjunction) connective $\circ$. Fusion stands to relevant $\rightarrow$ as *extensional* conjunction $\wedge$ stands classically to *material* implication, for which we use $\supset$ here. That is, we may contrast

(1) $A \circ B$ *entails* C iff A *entails* $B \rightarrow C$ (Relevant deduction theorem).
(2) $A \wedge B$ *entails* C iff A *entails* $B \supset C$ (Standard deduction theorem).

Put in a way to gladden algebraists' hearts, $\rightarrow$ is the *residual* of $\circ$, as classical (or intuitionist) $\supset$ is the residual of $\wedge$.

We have just seen that *full* relevant logics like **E** have *both* a relevant implication $\rightarrow$ and an extensional conjunction $\wedge$. But where is their *fusion* $\circ$? One might ask the same question of the relevant logic **R**. Through a happy syntactical accident, fusion is in fact *definable* in **R**, which Dunn invoked in [11] to produce the class of *DeMorgan monoids*. These are algebraic counterparts of **R** as *Boolean algebras* are counterparts of classical propositional logic. It then turns out, applying semantic insights as in [24], that fusion (which is anyway *wanted*) belongs and may be incorporated in *every* relevant logic.

We have, at this point, *two* formal analogues of *and*, an extensional $\wedge$ and an intensional $\circ$. This is the sort of thing that has led Lewis [18] to the complaint that relevant logics are *logics for equivocators*. We answer that to make this complaint is to misunderstand the *utility* of formal methods. Logicians *model* arguments in much the same way as physicists devise and apply mathematical systems to model the World. Our job, whether as logicians or physicists, is both to be faithful to the *data* of our trade and to formulate the *laws* of that trade with as much care as our subject requires.

In Physics, this dual obligation leads on occasion to *painful* revision. Theories accepted everywhere bump into *hard* facts. Take some simple, satisfying law: we pick $pv = nRT$ (from high school physics), on which the *volume* of an ideal gas increases with increasing *temperature* but decreases with increasing *pressure*. So it does, no doubt, but the law states an *exact* mathematical relationship. The first problem is that real gases are not ideal. The problem bites because under special conditions—very low temperatures, for one—this relationship becomes more and more *inexact*. And so the nice and memorable Boyle's law above gives way to (e.g.) van der Waals' equation, which is more complex but less inaccurate (until it too breaks down).

Something like this is what *classical logic* has done to us. Its simplicities bring joy. (*Everyone* comes to *comprehend* truth tables.) But its oversimplifications bring agony. (Many of the traditional problems in Philosophy of Science are pseudo-problems induced by what was naively taken to be Our Logic.) If a *false A* may be taken with Russell to imply everything, why has there been a *problem* of counterfactual conditionals? And why will the student *fail* her history examination if she says that the *reason* for the outbreak of World War I is that sugar is sweet? "I have given you," she may insist, "a true antecedent of what logicians say is a true conditional. If that does *not* count as a reason, what does?"

But let us not dwell further on the straightforward *contradiction by the data* of a favored theory. If Logic were Physics, van der Waals (to say nothing of Einstein and Bohr) would long since have revised the theory to take better account of the data. And this returns us to *bunches*, and to how premisses are put together for the sake of argument. The extensional intuition, built into ordinary $\wedge$, is that it suffices to use *one* of the A's in $A_1 \wedge \cdots \wedge A_n$ to have a valid argument. The relevant intuition, appropriate to fusion $\circ$, is that there must be a way of using *all* of the A's in $A_1 \circ \cdots \circ A_n$ to have a valid argument. For

When A and B have been fused,

Both A and B must be used.

We commit no offense against *reason* (and certainly do not *equivocate*) in having for theoretical reasons 2 *distinct* formal counterparts of *and*, each subject to *its own* laws, in our formal logic. And we may now, adapting for the general case the Gentzen-style analyses of **R**+ of [12,26], *bunch our premisses* as we will, to any desired depth of nesting, taking for granted only those *principles of collection* that a logic *enforces* for the mode of conjoining in question.

2. Relevant semantics

Yes, but what do these relevant logics like **R** and **E** amount to *semantically?* This was for a while a seriously *open* question. [14,30,33] began to supply some answers.

2.1. A modal paradigm for unary $\Box$ and $\Diamond$

As I see it, the semantics of relevant logics generalizes on the plan that Kripke and others developed for the explication of *modal* logics. On the plan of [16] we have a collection K of *worlds*, related by a 2-place accessibility relation R. We may say that w *sees* w' just in case wRw'. The key Kripke *truth-conditions* on the 1-place modal operators $\Diamond$ and $\Box$ are then the following:

$T\Diamond$. $\Diamond A$ is *true* at w iff A is true at *some* w' that w sees.
$T\Box$. $\Box A$ is *true* at w iff A is true at *every* w' that w sees.

2.2. Applying the paradigm to relevant $\rightarrow$ and $\circ$

What is the plan for a *similar* explication of the (irreducibly) 2-place operator $\rightarrow$? The simplest (and most Tarskian) course is to introduce a *3-place* relation R. We need some English expression that will do for a *ternary* R what *sees* does for a binary R. So let us survey again those native insights that render *irrelevance* objectionable. While lots of things in the world are *connected*, various other things are *unconnected*.

The *logical* signal that A and B are connected is that the *implication* $A \rightarrow B$ is *true*. Without growing *too metaphysical* here, we remind readers that there is an *old story* on which the premisses of a good argument are *necessarily connected* to its conclusion. Modern modal logic sought to take account of the ingredient of *necessity* in this story. But the only immediately obvious *connection* was that offered by the material $\supset$, with all its indifference to real relations among propositions or in the world.[1]

We have told another story. When one wishes to state the rule of *modus ponens* for $\rightarrow$, there are many apparently equivalent ways of stating it. In (pigeon) Formalese, here are a few:[2]

(a) $\vdash A$ and $\vdash A \rightarrow B \supset\ \vdash B$.
(b) $\vdash A \rightarrow B$ and $\vdash A \supset\ \vdash B$.
(c) $\vdash A \supset (\vdash A \rightarrow B \supset\ \vdash B)$.
(d) $\vdash A \rightarrow B \supset (\vdash A \supset\ \vdash B)$.

[1] We reject, as noted above, the *notational imperialism* on which even the material conditional is so often symbolized by $\rightarrow$ these days. In this paper we follow Peano and Whitehead and Russell and (in his heyday) Quine in using $\supset$, leaving $\rightarrow$ for an *honest* implication.

[2] We use, here and henceforth, $\supset$ *metalogically* to express *rules*. Like $\rightarrow$, the convention on $\supset$ is that implicative particles associate to the *right*, with logical particles binding *more tightly* than metalogical ones. We use $\equiv$ for 2-sided rules. For binary particles the full precedence order is $\circ, \wedge, \vee, \rightarrow, \leqslant, \geqslant, \supset, \equiv$.

People sunk in classical lassitude will not notice much difference among (a)–(d). Recall, though, that traditionally logicians distinguished between the *major premiss* $A \to B$ of *modus ponens* and its *minor premiss* A. That is, *modus ponens* comes with a *direction*, with the major premiss taking us *from* the minor *to* the conclusion. I am accordingly inclined, these days, always to state the rule according to the rubric (d), dismissing (a) and (c) as lazy alternatives and feeling comfortable with (b) only if *and* means what $\circ$ *says* in Formalese.

Think initially of the *first* argument of our *ternary relation* R as a domain of necessary connections—more briefly, of *laws*. These do not have to be laws of *logic*—biologists, computer scientists, physicists and economists have as much responsibility for stating the laws of their subjects as logicians have for setting out theirs. Whatever the source of such laws, we take it to be the job of $\to$ statements to *express* them *syntactically*. They are *major premisses*. The *second* argument of R takes account of the initial conditions supplied to laws—briefly, of *input*. Nor do we assume in general that the input belongs to the *same* world (or theory) as the law. It is a commonplace, after all, that the job of *experiment* is to provide data to *confirm* (or perhaps to *refute*) candidate laws. It is a methodological *no–no* to confuse the laws with the data. The *third* argument of ternary R tallies the result of applying the laws to the input; in a nutshell, it is the *output*.

With those ideas in mind, let us give necessary connection a *direction* by reading ternary $Rwxy$ as

(1) w *directs* x *to* y.

Retaining the *worlds* metaphor, while dropping as in [31] any constraint that the worlds in question be *logical* or even *possible*, let us say, for any world w and formula A, that

(2) w is an A-world

if and only if

(3) A is true at w.

We now recall from [32] the key relevant *truth-conditions* on the 2-place implication connective $\to$ and the fusion connective $\circ$ discussed in Section 1.3 above. (cf. also, e.g., [30]).

$T\to$. $A \to C$ is true at w iff w directs *all* A-worlds to C-worlds.

$T\circ$. $A \circ C$ is true at w iff *there exists* an A-world that directs a C-world to w.

We introduce some notation henceforth to express truth-conditions like $T\to$ (which leads as in [23] to nice translations of relevant logics into standard first-order logic). We express formulas as *predicates* of worlds, and appeal to standard notational conventions. We get

$$T\to.\quad [A \to C]w = \forall a \forall c(Rwac \supset Aa \supset Cc),$$

$$T\circ.\quad [A \circ C]w = \exists a \exists c(Racw \wedge Aa \wedge Cc).$$

as succinct ways of expressing the above truth-conditions.

2.3. Extensional connectives interpreted truth-functionally

We will express truth-conditions on logical particles in accordance with the conventions just introduced. Extensional $\wedge$ and $\vee$ are *interpreted* as one expects.

$T\wedge$. $[A \wedge B]w = Aw \wedge Bw$.
$T\vee$. $[A \vee B]w = Aw \vee Bw$.

Handling *negation* is a little trickier. On the [31] semantics, the original relevant De-Morgan negation $\sim$ is interpreted via a (Polish) technical maneuver, using an auxiliary unary operation * on worlds. The resulting truth-condition is

$T\sim$. $[\sim A]w = \neg[Aw^*]$.

Also of interest is making *Boolean* negation $\neg$ explicit, satisfying

$T\neg$. $[\neg A]w = \neg[Aw]$.

We postpone for now any further discussion of negation.

3. Demythologized worlds are theories

We have promised not to grow too metaphysical. On the other hand, the modal paradigm enriched for a relevant framework provides a *worlds* semantics. It is time to cash out those worlds in less exorbitant terms—to *demythologize* them. Happily the ingredients for this task are at hand. For our completeness proofs were presented using *theories*, which will do as the worlds desired.

3.1. Theories defined

A theory is a set of statements that hangs together logically. Its time to put a little meat on those bones. We suppose, in the first place, that we have a *logic L*, which furnishes a *binary relation* of logical consequence. We will use $A \leqslant B$ to indicate that B follows from A, according to *logic L*.[3] Then the *least* that we can demand of an *L-theory X* is that it *respect* $\leqslant$. I.e.,

$\leqslant E$. $A \leqslant B \supset A \in X \supset B \in X$.

On the other hand, we *also* expect a theory X to respect logical conjunction $\wedge$. I.e.,

$\wedge I$. $A \in X$ and $B \in X \supset A \wedge B \in X$.

It has become traditional in work on the algebra and semantics of relevant logics to characterize X as an *L-theory* just in case it satisfies $\leqslant E$ and $\wedge I$ for all formulas A and B.

[3] Often $A \leqslant B$ is indicated by the *theoremhood in L* of $\vdash A \rightarrow B$. But there are other signals– that $A \vdash B$ holds in a matching Gentzen consecution calculus. We like the *algebraic* flavor of $A \leqslant B$. It saves some parentheses. But we *mean* by it here simply that $A \rightarrow B$ is a theorem (since *modus ponens* is a main theme).

3.2. Truth-functional expectations on theories

But not all theories are created equal. There are also the intended meanings of *other* logical particles, like '*or*' and '*not*'. Surely some preference is to be given to theories that also respect what we *intend* by these particles. For example, the following conditions are imposed on theory X by our *regulative ideals*, as expressed in the truth-conditions $T\wedge, T\vee$ and $T\neg$ in Section 2.3.

$\wedge EI$. $A \in X$ and $B \in X$ iff $A \wedge B \in X$.

$\vee EI$. $A \in X$ or $B \in X$ iff $A \vee B \in X$.

$\neg EI$. $\neg A \in X$ iff $A \notin X$.

These conditions fare *differently* in various relevant (and other) logics. $\wedge EI$ is more or less *built in*—first, by the imposition of $\wedge I$ on *all* theories; second, by the *standard* character of $A \wedge B \leqslant A$ and $A \wedge B \leqslant B$. The $\vee I$ half of $\vee EI$ likewise tends to be built in, since $A \leqslant A \vee B$ and $B \leqslant A \vee B$ are also standard. But $\vee E$ is more difficult to come by. [4] Some intricate technical maneuvers accompany this point in semantical completeness proofs, often involving the *distributive lattice* properties of the usual relevant logics. Finally, we turn to the care and feeding of negation, which is even more difficult to come by. In the long run, we prefer our theories to be *consistent* and *complete* with respect to negation. But in practice, we are usually *not yet* in the long run. On this point too, delicate results turn.

3.3. Modus ponens products

Let X, Y and Z be any sets of formulas—not further specified for the moment, but we are aiming at theories. Let us define on such formula sets the following operation $\circ$:

$D\circ$. $X \circ Y = \{B : \exists A(A \to B \in X \text{ and } A \in Y)\}$.

The operation $\circ$ has had various names. Powers introduced it in [27] as *modus ponens product*. Fine [14] called it *fusion*. But the intuition behind it is one that we have been driving at throughout this paper, especially since Section 2.2. A formula C *belongs to* $X \circ Y$ just in case there is some way of performing *modus ponens*, taking a major premiss $A \to C$ from X and a minor one A from Y, getting C by an $\to E$ move.

And now *here* is the *canonical relation* R, on theories X, Y, Z.

DR. $RXYZ =\text{df } X \circ Y \subseteq Z$.

That's it. The magical ternary relation arises, on theories, as a way of looking after *modus ponens*.

[4] After all, we often *assert* disjunctions *without* having picked a disjunct. Knowing her, Vandy will go to the movies tonight, or stay home. But we may *not* know her well enough to know *which*.

But why, you may ask, did we not define the fusion operation $\circ$ *directly* on theories, together with some truth-condition like the following?

$$T{\rightarrow}\circ.\ [A \rightarrow B]X \text{ iff } \forall Y([A]Y \supset [B](X \circ Y)).$$

$T{\rightarrow}\circ$ was (near enough) the *original* truth-condition on relevant $\rightarrow$, in Urquhart's *semilattice semantics* for $\mathbf{R}\rightarrow$ worked out in [33] and independently in unpublished work by Routley. Again independently, Fine [14] proposed $T{\rightarrow}\circ$ as a central ingredient in Fine's *operational-relational semantics* for relevant logics. It lurks in the background of the relational semantics. But I do not yet see a *smooth* way of moving it to the foreground.

3.4. The calculus of relevant theories

Let then L be a given (relevant, or even irrelevant) logic. Formulas A are built up as usual from propositional variables p under $\wedge, \vee, \rightarrow$ and perhaps other connectives and propositional constants. We use S for the set of all formulas. We presume the entailment relation $A \leqslant C$ on L indicated by $\vdash A \rightarrow C$ in L. We already defined L-theories via $\leqslant$-closure and $\wedge$-closure above. (Algebraists would call them filters, as Dunn [11] observed.)

We turn now to the *Calculus of L-theories* $\mathbf{CLT} = \langle CLT, \circ, \subseteq \rangle$, where CLT is the set of *all L-theories*, [5] $\circ$ is the binary *modus ponens product* (or *fusion*) operation defined on sets of formulas by $D\circ$ above and $\subseteq$ is the subset (here, subtheory) relation. It is normally a simple exercise, safely left to readers, to show that CLT is indeed *closed* under $\circ$.

4. Combinatory logic CL

In Section 2.2 we laid down truth-conditions for the properly relevant connectives $\rightarrow$ and $\circ$. But we have not yet said anything about the *semantical postulates* to be imposed on the 3-place relation that builds in the *direction* enjoined by *modus ponens*. Such postulates, applied to the 2-place relation that served as our *paradigm* in Section 2.1, enabled Kripke [16] and others to *characterize* various modal logics and to *distinguish* them from each other. What will serve as a guide to do the same for relevant logics?

Our answer is "Combinatory Logic" (henceforth, CL) in the sense of Curry. We give a quick recapitulation of central ideas. The *atoms* of CL consist of countably many *variables* (for which we use 'x', etc.) and some primitive *constants* (among them some selection among I, S, K, B, B', W, C, and perhaps others). The *terms* (or, following [9], the *obs*) are built up from atoms and constants using a single binary operation symbol, for which we use '$\circ$', interpreted as the *application* of a 1-place function to an argument. We use 'M', 'N', etc. for arbitrary obs.

The intuitive *universe* of CL consists of 1-place functions. But n-place functions may be *simulated* (or *curried*, though the original idea seems to have been Schönfinkel's) by

[5] One may (or may not) wish to count the null set Λ and the set S of all formulas as *L-theories*.

iterated application of 1-place functions. Parentheses are used for grouping subterms, subject to the conventions of (a) dropping '$\circ$' for simple juxtaposition and (b) associating to the *left*, for ease in reading terms. We deal here only with *pure* combinatory logic, building up its theory of (weak) equality.

We begin with axioms for 1-step *contraction*, for which we use '$>_1$'. Each constant is governed by an accompanying axiom (scheme) for $>_1$. Among the famous ones are

I. $IX >_1 X$,

K. $KXY >_1 X$,

K^*. $K^*XY >_1 Y$,

S. $SXYZ >_1 XZ(YZ)$,

B. $BXYZ >_1 X(YZ)$,

B'. $B'XYZ >_1 Y(XZ)$,

W. $WXY >_1 XYY$,

C. $CXYZ >_1 XZY$,

C^*. $C^*XY >_1 YX$,

W^*. $W^*X >_1 XX$.

That is enough for now. We will use $\geqslant$ for the reflexive, transitive closure of $>_1$, imposing the additional monotonic principles

μ. $X \geqslant Y \supset ZX \geqslant ZY$,

ν. $X \geqslant Y \supset XZ \geqslant YZ$.

Suppose that $X \geqslant Y$ in CL. We then say that X *contracts* (or *reduces*) *to* Y, and that Y *expands to* X. Similarly we may say that Y is a *contraction* of X, and that X is an *expansion* of Y.

The *aim* of CL is to develop a theory of the *equality* of functions. So we introduce a further predicate '$=$', which is the symmetric and transitive closure of $\geqslant$. Central to the epitheory of CL is however the following theorem, stated in [9] as

Church–Rosser theorem. *Suppose* $X = Y$. *Then there is a* Z *such that* $X \geqslant Z$ *and* $Y \geqslant Z$.

This theorem establishes the *consistency* of CL. So there is a central sense in which the CL theory of *equality* is grounded and secured in its theory of *reduction*.

Inspired by Powers [27], I devised around 1970 the *Fools Model* (*FMO*) of Combinatory Logic. (cf. [22].) Here, all the combinators are simply taken as the closure under substitution of their "Curry types"—i.e., sets of pure $\rightarrow$ formulas. Application is *simulated* by the fusion operation $\circ$ defined in Section 3.3 by $D\circ$, on *arbitrary* sets of $\to$ formulas. We write, temporarily, '$\forall[A]$' to indicate the set of all substitution instances of a formula scheme A. Among the definitions were these of some famous combinators.

4.1. Some FMO "combinators"

$$I = \forall[p \to p],$$

$$C = \forall[(p \to q \to r) \to q \to p \to r],$$

$$B = \forall[(q \to r) \to (p \to q) \to p \to r],$$

$$C^* = \forall[p \to (p \to q) \to q],$$

$$B' = \forall[(p \to q) \to (q \to r) \to p \to r],$$

$$K = \forall[p \to q \to p],$$

$$K^* = \forall[p \to q \to q],$$

$$W = \forall[(p \to p \to q) \to p \to q],$$

$$S = \forall[(p \to q \to r) \to (p \to q) \to p \to r],$$

4.2. FMO sometimes models combinatory equality

These are among the identities that *FMO* verifies, where X, Y, Z are *any* sets of $\to$ formulas: [6]

$$IX = X,$$

$$CXYZ = XZY,$$

$$BXYZ = X(YZ),$$

$$C^*XY = YX,$$

$$B'XYZ = Y(XZ).$$

These correspond, in the vocabulary of Combinatory Logic, to 1-step *reductions*. We saw above that (weak) combinatory *equality* is the reflexive, transitive, symmetric and monotonic *closure* of 1-step reduction. We define *BCI-combinators* inductively by
(1) B, C, and I are BCI-combinators,
(2) If M and N are BCI-combinators, so is $M \circ N$.
Still adapting Curry, let us extend this definition by calling a term M a *BCI-ob* by adding
(3) If M is a variable or a BCI-combinator then M is a BCI-ob,
(4) If M and N are BCI-obs, so is $M \circ N$.

[6] Apply now the usual conventions of **CL**, dropping $\circ$ for juxtaposition and associating to the *left*.

We have then

BCI equality fact. Let the BCI obs M and N be demonstrably weakly equal in CL. Then also $v(M) = v(N)$ in FMO, where v is any assignment of sets of $\rightarrow$-formulas to variables which respects $\circ$ and the fixed interpretation of the BCI-combinators. The BCI-fact will hold moreover for any combinator *definable* from B, C, I (e.g., as B' is definable as CB).

4.3. FMO always models combinatory reduction

Some of the other laws, sadly, only hold in the *reduction* direction. E.g., we have

$$KXY \subseteq X,$$

$$K^*XY \subseteq Y,$$

$$WXY \subseteq XYY,$$

$$SXYZ \subseteq XZ(YZ).$$

Particularly disturbing is the K case, which *almost* holds as an equality. But $KX\Lambda = \Lambda$, where Λ is the *empty set* of formulas. Could we not *rule out* Λ? Alas no, since $WI = \Lambda$, famously.[7] (Still, there is an extension of the BCI-fact above to a corresponding BCK-fact, for *non-empty* BCK obs.) But it is nonetheless true that if any combinatory ob M (weakly) *reduces to* a term N, then as in the examples above the corresponding *subset* relation will surely hold in *FMO*.[8]

Specifically, a *combinatory ob* is defined inductively as follows:

(4) If M is a variable then M is a combinatory ob,

(5) If M is $S, K, B, C, I, W, B', C^*$ or K^* then M is a combinatory ob,

(6) If M and N are combinatory obs, then so is $M \circ N$.

Combinatory reduction fact. Let M and N be combinatory obs, and let $M \geqslant N$. Then also $v(M) \subseteq v(N)$ in FMO, where v is any assignment of sets of $\rightarrow$-formulas to variables which respects $\circ$ and the fixed interpretation above of the combinators.

The choice of primitive combinators under (5) is more or less up to us. S and K will do. We give a *long* list, since different choices of primitive combinators yield different logics.

5. The key to the universe

The above plot thickens in the presence of logical particles besides $\rightarrow$. And (of all things) our modeling of **CL** *improves*.

[7] That is, on the Curry and Feys [9] analysis, WI "has no type". We look further at WI below.

[8] In virtue, if you please, of the Subject Reduction Theorem of Curry and Feys [9].

5.1. Modeling I

A logical axiom scheme *almost always* [9] present is

AxI. $\vdash A \to A$

Let now **I** denote the closure of all instances of AxI under $\leqslant E$, $\wedge I$, where $\leqslant$ is the *entailment* of the minimal positive relevant logic **B**+ of Routley and Meyer [30]. [10] Amazingly, **I** so defined *coincides* with the set of theorems of **B**+. Note also, **CL** and $\boldsymbol{\lambda}$ fans, that **I** so defined has *exactly the right properties* to mimic the *combinator* I. For

I fact. Let X be *any* **B**+-theory. Then $\mathbf{I} \circ X = X$.

Proof. The l. to *r*. inclusion holds because **I** *is* **B**+, and **B**+-theories are closed under **B**+-entailment. The r. to l. inclusion holds on the reasoning that yields $X \subseteq I \circ X$ in Section 4.2.

5.2. Modeling other combinators

Even readers who know relevant and other substructural logics well may be *insufficiently impressed* with the I fact. After all, we saw in the BCI equality fact in Section 4.2 that even the Fools model FMO captures I perfectly. This is more than we can say for cancellators like K and K^*, and a lot more than we can say about combinators with a duplicating effect like W and S.

Let us begin with the duplicators. The most depressing fact in the Fools model FMO is that the combinator WI (which we henceforth follow Curry and Feys [9] in abbreviating W^*) *ought to* produce

$$W^* \circ X = X \circ X,$$

a *pure* duplication, but instead $W^* = \Lambda$ and then $W^* \circ X$ is the null set Λ. But there is a remedy for this depression. It arises because, in FMO,

$$W^* = W \circ I = \forall[(p \to (p \to q)) \to (p \to q)] \circ \forall[r \to r]$$

and there is no way to find a common substitution instance $A \to (A \to B)$ of an *antecedent* of W and $C \to C$ of I. In the parlance of the automated reasoners, we cannot *unify* $p \to (p \to q)$ and $r \to r$. For the cost of so doing would be to *identify* a formula $A \to B$ with its own proper subformula A. That's why $W^* = \Lambda$ in the Fools model.

Our preoccupation with *theories* rides now to the rescue. In FMO, the "combinators" were sets of $\to$ formulas, closed under substitution. Suppose instead that we pursue further the policy in Section 5.1 by making *all* the combinators into theories, requiring

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[9] Not *quite* always. Martin [19] gives a nice solution to Belnap's **P-W** problem, on which $A \to A$ is an *anti-theorem* of $\mathbf{S}_{\to}$ *none* of whose instances are provable from AxB, AxB' and $\to E$.

[10] We take **B**+ here as formulated in the $\to, \wedge, \vee$ vocabulary, with optional extras to taste.

also closure under both $\wedge I$ and an appropriate relevant entailment relation $\leqslant$. This will do wonders for W^*. For since $(A \to B) \leqslant A \to B$ in $B+$, we have in $B+$, on a couple of steps of antecedent replacement,

(1) $(A \to B) \wedge A \leqslant (A \to B) \wedge A \to B$.

For ease in reading the formula, set

(2) $\alpha = (A \to B) \wedge A$.

We observe on simple FMO principles,

(3) $(\alpha \to (\alpha \to B)) \to (\alpha \to B) \in W$

while the coincidence between $B+$ entailment and membership in the combinator **I** yields by (1)

(4) $\alpha \to (\alpha \to B) \in I$

whence by (3) and (4) and *modus ponens*

(5) $\alpha \to B \in W \circ I$.

I.e., applying (2) to (5),

(6) $(A \to B) \wedge A \to B \in W^*$.

We must now pause for some *annoying* technicalities. The reasoning just gone through means that we have found a "type" for the combinator W^* (which was more than Curry and Feys [9] could do). Ronchi della Rocca and Venneri [29] shows that we have done better. We have found a *principal type* for W^*, which we set down as

$\mathbf{W}^* = \forall[(p \to q) \wedge p \to q\}$,

where the schematic notation $\forall[\ldots\}$ now indicates that the displayed formula is closed under *all* of uniform substitution, $\wedge I$ and $\leqslant E$, where $\leqslant$ is supplied by the minimal relevant logic. Having handled **I** already as $\forall[p \to p\}$, we expand the schemes above by likewise laying it down that

$\mathbf{C} = \forall[(p \to q \to r) \to q \to p \to r\}$,

$\mathbf{B} = \forall[(q \to r) \to (p \to q) \to p \to r\}$

and similarly for "combinators" $\mathbf{C}^*, \mathbf{B}'$, $\mathbf{W}, \mathbf{S}$, etc.

Before tackling the cancellators K and K^*, we pause for the first of our annoying technicalities. Back in FMO we observed that the *reduction* direction of modeling CL was the proverbial *piece of cake*. The *expansion* direction, though it was perfect on BCI-obs, just *failed* on the cancellators and duplicators. As it now turns out, it is *expansion* which is the piece of cake, while reduction is no longer so pleasant. We illustrate with $\mathbf{W}^*$.

Show $\mathbf{W}^* \circ X = X \circ X$, where X is a theory:

Expansion: Suppose $B \in X \circ X$. For some A, $A \to B \in X$ and $A \in X$, by $D\circ$. But then, by $\wedge I$, $(A \to B) \wedge A \in X$. Because $(A \to B) \wedge A \to B \in \mathbf{W}^*$, we have by $D\circ$ that $B \in \mathbf{W}^* \circ X$.

Contraction: Suppose $B \in \mathbf{W}^* \circ X$. Then there is some $C \to B \in \mathbf{W}^*$ such that $C \in X$. How in the world does this ensure that B is in $X \circ X$?

There will be an answer (as the song assures us). But it is an answer *sensitive to vocabulary*, enjoined by Dezani's beautiful *Bubbling lemma* (2.4(ii) of Barendregt et al. [4]). Meanwhile we look after the cancellators. We begin with the *theories*,

$$\mathbf{K} = \forall[p \to (q \to p)\},$$

$$\mathbf{K}^* = \forall[p \to (q \to q)\}.$$

Our first aim is to show, for theories X and Y, that $\mathbf{K} \circ X \circ Y = X$. We know from our musings about FMO that we are in trouble if $Y = \Lambda$. Coppo and his colleagues found a neat way out of this trouble. (cf. [4]). We will think of it as follows. The condition $\wedge I$ on theories assures that, where X is any non-empty set of formulas each of which belongs to the theory Y, so also must the *conjunction* $\wedge X$ of these formulas belong to Y (e.g., if $\{A,B,C\} \subseteq Y$ then $A \wedge B \wedge C \in Y$). Let us extend this thinking to the empty set Λ. Lattice-theoretically, the *meet* (i.e., conjunction) of the empty subset of elements of a lattice L is conveniently computed as the **Top** element of L, which we call $\mathbf{T}$ ([4] called it ω, cf. [10].)

Intuitively, $\mathbf{T}$ will express the proposition that is true at *every* world. ($\mathbf{T}$ is a *Church* constant. Think of it with [3] as the trivially true *disjunction* $\exists p p$ of all propositions, to be contrasted with the $\mathbf{I}$-surrogate *Ackermann* constant $\mathbf{t}$, the more interestingly true *conjunction* $\forall p(p \to p)$). It is consistent with other minimal relevant ideas to lay down both $A \leqslant \mathbf{T}$ and $\mathbf{T} \leqslant A \to \mathbf{T}$, which as in [4] we henceforth assume.

A $\mathbf{T}$-theory is now constrained to contain $\mathbf{T}$, and to be closed as just suggested. Now we can make short work of (half of) $\mathbf{K}$.

K-*expansion*: Show $X \subseteq \mathbf{K} \circ X \circ Y$, where X and Y are $\mathbf{T}$-theories. Suppose $A \in X$. Anyway, $A \to (\mathbf{T} \to A) \in \mathbf{K}$, whence $\mathbf{T} \to A \in K \circ X$. But Y is a $\mathbf{T}$-theory, whence $A \in K \circ X \circ Y$. Done!

5.3. The fools model perfected in $B \wedge T$-theories

In this section we zero in on a fragment of the minimal relevant logic, which will exactly model weak equality in Combinatory Logic. $\mathbf{B}\wedge$ (pronounced *BAND*) will be the fragment of $\mathbf{B}+$ in just $\to$ and $\wedge$, given by the following axiom and rule schemes:

AxI. $\quad A \leqslant A$

Ax $\wedge E$. $\quad A \wedge B \leqslant A \quad$ and $\quad A \wedge B \leqslant B$

Ax$\to\wedge I$. $\quad (A \to B) \wedge (A \to C) \leqslant A \to B \wedge C$

Ru$\to$*E*. $A \leqslant C \supset \vdash A \supset \vdash C$

Ru$\wedge$*I*. $\vdash A$ and $\vdash B \supset \vdash A \wedge B$

Ru*B* $B \leqslant C \supset A \to B \leqslant A \to C$

RuB'. $A \leqslant B \supset B \to C \leqslant A \to C$

We have seen that, to get past our difficulties with the cancellators, it is advisable to throw $\mathbf{T}$ into the vocabulary, extending $\mathbf{B}\wedge$ to the richer system $\mathbf{B}\wedge\mathbf{T}$ (pronounced *BAT*). Add to the above

Ax$\mathbf{T}$*I*. $A \leqslant \mathbf{T}$

Ax$\mathbf{T}$*E*. $\mathbf{T} \leqslant A \to \mathbf{T}$

Note that Ax$\mathbf{T}$*E* becomes redundant if we have a fusion *connective*. Apply (1) in Section 1.3 to $\mathbf{T} \circ A \leqslant \mathbf{T}$.

We now present the Fools Model Updated (henceforth FMU) in the $\mathbf{T}$-theories of $\mathbf{B}\wedge\mathbf{T}$. As the domain of FMU we take the set of all $\mathbf{T}$-theories. Readers may enjoy themselves showing that FMU is closed under the fusion of $\mathbf{T}$-theories defined by $D\circ$. Expansion is the promised cake.

Combinatory expansion fact for FMU. *Let M and N be combinatory obs such that M weakly reduces to N. Then also $v(M) \supseteq v(N)$ in FMU, where v is any assignment of* $\mathbf{T}$*-theories to variables which respects $\circ$ and the fixed interpretation above of the combinators.*

Proof. Assign the $\mathbf{T}$-theory $\mathbf{K}$ above to the combinator K, and similarly for other combinators. Proceed as in the $\mathbf{W}^*$ and $\mathbf{K}$ expansion arguments to show that if $XY_1 \dots Y_n = Z$ results from 1-step reduction on a combinator X, then $\mathbf{X} \circ v(Y_1) \circ \cdots \circ v(Y_n) \supseteq v(Z)$. In general all that will be required for this verification is the "principal type" of the combinator in question, as above. The rest of the proof goes through inductively on the observation that $\supseteq$ is reflexive and transitive, while satisfying the monotonicity conditions, given $X \supseteq Y$, that $Z \circ X \supseteq Z \circ Y$ and $X \circ Z \supseteq Y \circ Z$. $\square$

The party is *over*, as we saw above, when it comes to modeling *contractions*. So we had better pause for the

Bubbling lemma (Barendregt et al. [4]). *Suppose $(A_1 \to C_1) \wedge \cdots \wedge (A_n \to C_n) \leqslant A \to C$ in* $\mathbf{B}\wedge$, *which we write as $\bigwedge_{i \in I}(A_i \to C_i) \leqslant A \to C$. Then there is some non-empty finite subset $J \subseteq I$ such that $A \leqslant \bigwedge_{j \in J} A_j$ and $\bigwedge_{j \in J} C_j \leqslant C$ in* $\mathbf{B}\wedge$.

We will now apply the Bubbling Lemma (henceforth, BL) to finish sketching a proof that the contraction half of our argument *ad* $\mathbf{W}^* \circ X = X \circ X$ is ok in FMU. Assuming $B \in \mathbf{W}^* \circ X$, we reached in Section 5.2 above the conclusion that there is a $C \to B$ in $\mathbf{W}^*$ such that $C \in X$. To be shown is that $B \in X \circ X$. We must now look a little more carefully at the members of $\mathbf{W}^*$. Analysis indicates that $D \in \mathbf{W}^*$ iff there is some conjunction of formulas of the form $(E \to F) \wedge E \to F$ which entails D in $\mathbf{B} \wedge \mathbf{T}$. D is in this case a particular formula $C \to B$. Omitting the feeding and care of $\mathbf{T}$, we get by BL, for some finite set J of indices, $C \leqslant \bigwedge_{j \in J} ((E_j \to F_j) \wedge E_j)$ while $\bigwedge_{j \in J} F_j \leqslant B$. Because $C \in X$, we have by closure of X under $\leqslant$ that each of the $E_j \to F_j$ is in X, while so also is each of the E_j in X. So, for all $j \in J$ we have F_j in $X \circ X$, by $D\circ$. By $\wedge I$ the conjunction of these F_j is in the theory $X \circ X$. But we just saw that this conjunction entails B. So B is in $X \circ X$, as desired.

So it goes, as the reader may verify by consulting [4,10]. The result is

Combinatory contraction fact for FMU. *Let M and N be combinatory obs such that M (weakly) reduces to N. Then also $v(M) \subseteq v(N)$ in FMU, where v is any assignment of* **T***-theories to variables which respects $\circ$ and the fixed interpretation above of the combinators.*

Proof. The hard part in each case is verifying the 1-step reduction principles for particular combinators. Leaving that to readers (which they may look up if necessary, or apply BL case by case for extra fun), the remaining inductive steps are straightforward as before. Done! □

We can now recapitulate the promised

Key2U theorem. FMU models CL. *Suppose that $M = N$ in* **CL**. Always $v(M) = v(N)$ *in FMU.*

Proof. By the contraction and expansion facts for FMU. Done! □

6. Semantical steps forward

We came in the last section to a rewarding realization—namely, that the semantics of relevant logics supplies, in its partnership with **CL** and $\boldsymbol{\lambda}$, the veritable key to the universe. Before being *too* carried away, let us pause to see what we have unlocked. We were concerned a while ago to relate **CL** to semantical postulate sets for *particular* relevant logics. It all had something to do with *bunches*, you may recall.

6.1. Combinators and bunches

We have contrasted an *extensional* bunching, under $\wedge$, with a *relevant* one, under $\circ$. But hitherto $\circ$ has only made an occasional appearance in this paper, and that principally not as a logical particle but as the metalogical operator defined by $D\circ$ in Section 3.3 on whole theories. Still, the metalogical operator can be parent to a logical one, since we set down in Section 2.2 a truth-condition To for evaluating *formulas* $A \circ B$ at *worlds* (or *theories*) w.

On the other hand, we have firm intuitions about how $\wedge$ should behave. We expect $\wedge$ to be associative, commutative and idempotent. It does *not* disappoint. What should we expect of $\circ$? That depends. In a *strong* relevant logic like **R**, fusion $\circ$ is indeed associative and commutative; it even delivers *semi*-idempotence, since $A \leqslant A \circ A$ in **R**. (The extension **RM** of **R** also provides the converse—at the cost, alas, of fallacies of relevance.) But weaker logics like **E** and **T** chop away at those smooth properties of $\circ$, until none of them are left in the minimal system **B** of Routley et al. [32].

I say that this is as it should be. For a choice among logics is at the same time a choice among *combinators*. To like the *prefixing axiom* $(B \to C) \to (A \to B) \to A \to C$ is, in the presence of an explicit fusion $\circ$, to affirm also one direction of the associativity of $\circ$. Specifically it is to affirm that $(D \circ E) \circ F$ logically entails $D \circ (E \circ F)$, in conformity to the 1-step reduction postulate for the matching combinator B in Section 4.

This behavior is ubiquitous. We can illustrate it at the level of $\mathbf{B}\wedge$, formulated explicitly in Section 5.3. We extend $\mathbf{B}\wedge$ to a system $\mathbf{B}[\to, \wedge, \circ]$, which makes fusion explicit and adds to the formulation above the *2-sided* residuation rule expected from our discussion in Section 1.3:

Ru$\to\circ$. $A \circ B \leqslant C \equiv A \leqslant B \to C$.

We shall now revert to using simple juxtaposition for fusion $\circ$, associating iterated fusions *to the left*. Iterated $\to$'s continue to associate *to the right*. It is straightforward to show that the following are theorems and derivable rules.

$B1$. $(A \to B)A \leqslant B$,

$B2$. $A \leqslant B \to AB$,

$B3$. $A \leqslant B \supset AC \leqslant BC$,

$B4$. $A \leqslant B \supset CA \leqslant CB$.

We now pin down the syntactic correlate of the Key to the Universe in the table below.

CL ob	Fusion fact	Implication fact
W^*	$A \leqslant AA$	$(A \to B) \wedge A \leqslant B$
C^*	$AB \leqslant BA$	$A \leqslant (A \to B) \to B$
C	$ABC \leqslant ACB$	$A \to (B \to C) \leqslant B \to (A \to C)$
W	$AB \leqslant ABB$	$A \to (A \to B) \leqslant A \to B$
B	$ABC \leqslant A(BC)$	$B \to C \leqslant (A \to B) \to (A \to C)$
B'	$ABC \leqslant B(AC)$	$A \to B \leqslant (B \to C) \to (A \to C)$
S	$ABC \leqslant AC(BC)$	$A \to (B \to C) \leqslant (A \to B) \to (A \to C)$
WB	$AB \leqslant A(AB)$	$(B \to C) \wedge (A \to B) \leqslant A \to C$
K	$AB \leqslant A$	$A \leqslant B \to A$
K^*	$AB \leqslant B$	$A \leqslant B \to B$

Combinator Correspondence Theorem (CCT). *Let* **L** *be any logic extending* $\mathbf{B}[\to, \wedge, \circ]$. *Then any of the fusion facts is a theorem scheme of* **L** *iff the corresponding implication fact is a theorem scheme.*

Proof. We do a couple of cases, leaving the rest as exercises to the reader.

Ad K, observe simply that $AB \leqslant A$ iff $A \leqslant B \to A$ in **L**, applying Ru $\to \circ$ in each direction.

Ad S. ($\Rightarrow$)

1. $(A \to B \to C) \circ (A \to B) \circ A$ $\leqslant ((A \to (B \to C)) \circ A) \circ ((A \to B) \circ A)$	S fusion fact,
2. $((A \to (B \to C)) \circ A) \leqslant B \to C$	$B1$,
3. $((A \to B) \circ A) \leqslant B$	$B1$ again,
4. $((A \to (B \to C)) \circ A) \circ ((A \to B) \circ A) \leqslant (B \to C) \circ B$	$2, 3, B3, B4$,
5. $(B \to C) \circ B \leqslant C$	$B1$,
5. $(A \to B \to C) \circ (A \to B) \circ A \leqslant (B \to C) \circ B \leqslant C$	$1, 4, 5$, $\leqslant$Transitivity,
6. $A \to (B \to C) \leqslant (A \to B) \to (A \to C)$	5, Ru$\to\circ$ (twice).

($\Leftarrow$)

Set $\alpha = AC(BC)$

1. $C \to (BC \to \alpha) \leqslant (C \to BC) \to (C \to \alpha)$	S Implication fact,
2. $B \leqslant C \to BC$	$B2$,
3. $C \to (BC \to \alpha) \leqslant B \to (C \to \alpha)$	1, 2, Monotonicity,
4. $AC(BC) \leqslant \alpha$	Dfα, $\leqslant$Reflexivity,
5. $A \leqslant C \to (BC \to \alpha)$	4, Ru$\to\circ$ (twice),
6. $A \leqslant B \to (C \to \alpha)$	5, 3, $\leqslant$Transitivity,
7. $ABC \leqslant AC(BC)$	6, Ru$\to\circ$ (twice), Dfα.

6.2. Implication on its head

Recall the truth-condition $T\to$ in the relational semantics

$$T\to.\ [A \to C]w = \forall a \forall c(Rwac \supset Aa \supset Cc).$$

What happens if we replace it with this truth-condition $T\to r$?

$$T\to r.\ [A \to rC]w = \forall a \forall c(Rawc \supset Aa \supset Cc).$$

The answer is, "It depends on the logic." In our hitherto paradigmatic logic **R**, nothing happens. (Neither does anything happen in the linear logic **LL** of Girard [15].) Put otherwise, $\to$ and $\to r$ are the *same* implication connective in **R**. But **R** is quite a strong relevant logic. But we have been lately thinking about *minimal* logics like **B**, from which the stronger logics arise by *imposing* special assumptions. And the *equivalence* of $Rwac$ and $Rawc$, which justifies on the [31] semantics the identification of $\to$ and $\to r$ for **R**, certainly looks like a special assumption.

This is true, as we noted in [24]. [11] If desired, we may add $\to r$ conservatively to minimal relevant vocabulary, together with the rule,

$$\text{Ru}\to r\circ.\ A \circ B \leqslant C \equiv B \leqslant A \to rC$$

[11] Dunn [13], though generous in acknowledging indebtedness to [24], misses this point. In fairness, it was Lambek who introduced $\to r$ to substructural logics. cf. [17].

Having $\rightarrow r$ in the vocabulary brings back at even the minimal **B**+ level some characteristic theorems of **R** and **LL**, by mixing and matching the $\rightarrow$'s. For example, consider

$C^*r\text{l}$. $A \leqslant (A \rightarrow r\, B) \rightarrow B$,

$C^*\text{l}r$. $A \leqslant (A \rightarrow B) \rightarrow r\, B$.

These variants of the C^* *theorem* of **LL** and of **R** are *already* provable at the *minimal* level.

What is the *use* of this twisted implication $\rightarrow r$? To begin with, we might reverse all the intuitions to which we have appealed so far, thinking of $\rightarrow r$ as the *native* implication and coming to prefer (c) in Section 2.2 as a preferential way of stating *modus ponens*. Thinking of implication (as we have) as resting on real relations between input and output, there is a neat and appealing symmetry in $T\rightarrow r$—input a on the *left*, output c on the *right* and (implicative) relation w in the *middle*.

Speaking personally (and yearning as my faithful readers know to be *traditional* in all things), I will leave my own intuitions as they are, sticking with Aristotle on the point that major premisses come *first*, as in Section 2.2(d). I also resist arrows pointing every which way, not wishing to confuse myself (and possibly others) by introducing $\leftarrow$ as a counterpart to $\rightarrow$ and setting it down that one of them shall be *left* and the other *right*. In my vocabulary, $\rightarrow$ is here a *left* residual, satisfying Ru$\rightarrow\circ$. And what I call the right residual $\rightarrow r$ is what satisfies Ru$\rightarrow r\circ$.

That being settled, what should we make of $\rightarrow r$? It is good, I think, to agree with Restall that the connectives $\circ, \rightarrow$ and $\rightarrow r$ form a *family*, with $\circ$ as the *parent* [28, p. 30]. And since our logics, when we sought to plumb their depths, forced $\circ$ upon us (e. g., in [24]) to account algebraically and semantically for $\rightarrow$, it would seem that in the general case they have no less forced $\rightarrow r$ upon us. We shall not, however, explore this line any further here. See [13,28].

6.3. *The classical minimal relevant logic* **CB**

The minimal logic **B** of Routley and Meyer [32] is the (conservative) result of adding a DeMorgan negation $\sim$ to **B**+, satisfying the truth-condition $T\sim$ in Section 2.3. More interesting, for present purposes, is the (still conservative) result of adding the fully Boolean negation $\neg$, governed by $T\neg$ in Section 2.3. The resulting system is **CB** (introduced as **CB**+ in [21]). We formulate **CB** here with $\circ, \rightarrow, \wedge, \neg$ primitive, subject to the following definitions.

$D\vee$. $A \vee B =\text{df}\ \neg(\neg A \wedge \neg B)$,

$D\supset$. $A \supset B =\text{df}\ \neg(A \wedge \neg B)$,

$D\equiv$. $A \equiv B =\text{df}\ (A \supset B) \wedge (B \supset A)$,

$D{\leftrightarrow}$. $A \leftrightarrow B =$df $(A \to B) \wedge (B \to A)$,

$D\mathbf{F}$. $\mathbf{F} =$df $p \wedge \neg p$, where p is the first propositional variable,

$D\mathbf{T}$. $\mathbf{T} =$df $\neg\mathbf{F}$.

We state quickly the semantics for **CB**. A CB model structure (CBms) is a triple $\mathbf{K} = \langle 0, K, R\rangle$, where K is a set (of worlds, if you like), $0 \in K$ (the real world) and R is a ternary relation on K. There is only 1 postulate, for all $a, b \in K$:

$p0$. $R0ab$ iff $a = b$.

Setting $\mathbf{2} = \{1, 0\} =$ (true, false, if you like), a *possible interpretation* I of the language L in $\mathbf{K}$ is any function $I : L \times K \to \mathbf{2}$. Using again $[A]a$ for $I(A, a) = 1$, etc., I is moreover an *interpretation* of L in K if it satisfies the appropriate truth-conditions $T\neg, T\wedge, T{\to}, T\circ$ above. Note that the interpretation I will automatically satisfy appropriate truth-conditions $T\vee, T\supset$, etc., and that the (Boolean) conditions on truth-functions assure $I(\mathbf{T}) = 1$ and $I(\mathbf{F}) =$ false always. A formula A is *verified* on the interpretation I iff $I(A, 0) = 1$; it is *valid in* $\mathbf{K}$ iff it is verified on all interpretations therein; finally, A is **CB**-*valid* iff A is valid in every **CB**ms $\mathbf{K}$.

We may, as a first approximation, simply identify the *logic* **CB** with the set of **CB**-valid formulas. We note the following easy theorem:

Finite model theorem. *Every non-theorem of* **CB** *is invalid in some* finite *model.*

Proof. By adaptation of Routley's *filtration method* [32], as by Brady in [5, 277pp.]. Specifically, suppose B a non-theorem of **CB**. Then there is a **CB**ms $\mathbf{K} = \langle 0, K, R\rangle$ and an interpretation I in $\mathbf{K}$ such that *not* $[B]0$. If K is finite, we are through. So suppose K infinite. Let Sub$(B) = \{A$: A is a subformula of $B\}$. Sub(B), at least, is finite. We define an *equivalence relation* Q on K by setting, for all $x, y \in K$, xQy iff, for all $A \in$ Sub(B), $I(A, x) = I(A, y)$. Consider now the result K/Q of collapsing K modulo Q. At least K/Q is finite (since the finitely many subformulas of B can separate at most finitely many worlds). We refer to K/Q henceforth simply as K', and to the equivalence classes that are its members as a', etc. K' will be moreover a **CB**ms as soon as we provide it with a $0'$ and define a ternary relation R' on it. For the second of these tasks, let $R'a'b'c'$ hold iff *there exist* a in a', b in b' and c in c' such that $Rabc$. Then, *ignoring* $0/Q$ (*except as* one of the a'), we add a *new* $0'$ subject to the condition $p0$ on K', R'. Finally, use I to define an interpretation I' in $\mathbf{K}' = \langle 0', K', R'\rangle$ by setting, for each propositional variable p in Sub(B), (i) $I'(p, 0') = I(p, 0)$ and (ii) $I'(p, a') = I(p, a)$ for a in a' (recalling that the a *agree* on I on all subformulas of B). Extend I' uniquely to all relevant formulas by imposing $T\neg, T\wedge, T{\to}, T\circ$. We wish now to show that, for all A in Sub(B) both (1) $I'(A, 0') = I(A, 0)$ and (2) for all a in K, $I'(A, a') = I(A, a)$. (1) and (2) hold by stipulations (i) and (ii) when A is a variable. We turn to the inductive case. The conclusion is immediate on IH when A is of the form $\neg C$ or $C \wedge D$. Suppose that A is $C \to D$. Given $p0$, (1) reduces to $\forall x'([C]x' \supset [D]x')$ iff $\forall x([C]x \supset [D]x)$, which clearly holds on IH. (2) becomes $\forall y'\forall z'(R'a'y'z' \supset [C]y' \supset [D]z')$ iff $\forall y\forall z(Rayz \supset [C]y \supset [D]z)$. Here it is important that C and D both are subformulas of the subformula A of B, and are accordingly covered by the IH. Suppose first that A is false on I at a. Then by $T\!\rightarrow$ there are c and d such that $Racd$ and $[C]c$ but not $[D]d$. By definition of R' we have also $R'a'c'd'$. Combining this with the IH we get also $[C]c'$ but not $[D]d'$, refuting A at a' on I' by $T\!\rightarrow$. Suppose conversely that A is false on I' at a'. Then by $T\!\rightarrow$ there are c', d' such that $R'a'c'd'$ and $[C]c'$ but *not* $[D]d'$. By definition of R' there are a in a', c in c' and d in d' such that $Racd$, while $[C]c$ but *not* $[D]d$ on IH. So A is refuted also at a on our original interpretation I in K. This completes the verification of (2) and ends the inductive argument re $C \rightarrow D$. Finally consider the case where A is $C \circ D$. (1) now reduces by $T\circ$ to the claim that $\exists y'\exists z'(R'y'z'0' \wedge [C]y' \wedge [D]z')$ iff $\exists y\exists z(Ryz0 \wedge [C]y \wedge [D]z)$. It is time to appeal to the special properties of 0 and of $0'$, imposed by the fiat $p0$. The *only* possibility, given $p0$, to satisfy $Ryz0$ occurs if $y = z = 0$, and similarly for $R'y'z'0'$. So the claim that must be verified for (1) in this case is $[C]0' \wedge [D]0'$ iff $[C]0 \wedge [D]0$, and this is true on IH. (Caution: this gets *trickier* in the case of the $\mathbf{CR}^*$ of Meyer et al. [25], since $Ryz0$ may be satisfied *non-trivially* in that case. But filtrations do not work anyway for that *undecidable* system!) Finally we must verify (2) in the $C \circ D$ case, which boils down by $T\circ$ to $\exists y'\exists z'(R'y'z'a' \wedge [C]y' \wedge [D]z')$ iff $\exists y\exists z(Ryza \wedge [C]y \wedge [D]z)$. Argue as in the $\rightarrow$ case, assuming this time one side *true* on the associated interpretation and using the IH to show that the other side also must be true. In conclusion we have shown that the bad guy B takes the *same* value (namely *false*) at $0'$ in the chopped down finite **CB**ms on I' that it took in the infinite **CB**ms on I. End of proof! □

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